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(DEPARTMENT OF PURE MATHEMATICS)

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TWO NEW NEARLY KIRKMAN TRIPLE SYSTEMS

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# Two new Nearly Kirkman Triple Systems

by

A.E. Brouwer

ABSTRACT

Baker and Wilson constructed Nearly Kirkman Triple Systems NKTS(6t) = RDG(3,1,2;6t) for all $t \geq 3$ except $t = 14, 17, 29$. Here we give a direct construction of a NKTS(6t) for all odd t, $t \geq 11$. In particular this solves the cases $t = 17, 29$ so that $t = 14$ remains the last unsolved case.

KEY WORDS & PHRASES: *Nearly Kirkman Triple System, resolvable group divisible design.*

INTRODUCTION

A Nearly Kirkman Triple System NKTS(v) (nomenclature of KOTZIG & ROSA [3]) or a resolvable group divisible design RGD(3,1,2;v) (HANANI-like notation) is a resolvable pairwise balanced design on v points, with one parallel class consisting of pairs and the remaining parallel classes consisting of triples. Obviously a necessary condition for the existence of a NKTS(v) is $v \equiv 0 \pmod 6$ or $v = 2$, and KOTZIG & ROSA [3] showed that a NKTS(6t) does not exist for t = 1,2 while it does exist for t = 3. BAKER & WILSON [1] proved the existence of a NKTS(6t) for all $t \geq 3$ except for $t \in \{14,17,29\}$. In this note we give a direct construction of a NKTS(6t) for t odd, $t \geq 11$. This means that 6.14 = 84 is the only remaining order for which the existence of a NKTS(v) is not yet decided.

## The ingredients

We need some auxiliary designs:

(a) A NKTS(18), which can be found in [3] or constructed directly by doubling the affine plane AG(2,3).

(b) A group divisible design GD(4,1,6;6m) for all $m \geq 5$. These designs are constructed by BROUWER, HANANI & SCHRIJVER [2].

(c) A pairwise balanced design on a set V of 18 points with 6 parallel classes of triples and 1 block Z of size 6 and on the remaining 12 points (i.e. on V\Z) 2 parallel classes of triples and one parallel class of pairs.

[Note that indeed $\binom{18}{2} = 153 = 6.18 + \binom{6}{2} + 2.12 + 6$ so that this is possible numerically.]

This design was found by PDP11 and can be described as follows:

Let $V = Z \cup I_{12}$, where $I_{12} = \{0,1,\ldots,11\}$ and $Z = \{a,b,c,d,e,f\}$.

Take the parallel classes on V:

1. {a,0,11} , {b,1, 3} , {c,4, 8} , {d,6,10} , {e,2, 7} , {f,5, 9}
2. {a,7, 9} , {b,0,10} , {c,1, 5} , {d,2, 4} , {e,8,11} , {f,3, 6}
3. {a,4, 6} , {b,2,11} , {c,0, 9} , {d,5, 7} , {e,3,10} , {f,1, 8}

4. {a,1,10} , {b,5,6} , {c,2,3} , {d,0,8} , {e,4,9} , {f,7,11}
5. {a,3,8} , {b,4,7} , {c,6,11} , {d,1,9} , {e,0,5} , {f,2,10}
6. {a,2,5} , {b,8,9} , {c,7,10} , {d,3,11} , {e,1,6} , {f,0,4}

And the parallel classes on V/Z:

i. {0,1,2} , {3,4,5} , {6,7,8} , {9,10,11}
ii. {0,3,7} , {1,4,11} , {2,6,9} , {5,8,10}
I. {0,6} , {1,7} , {2,8} , {3,9} , {4,10} , {5,11}.

<u>The construction</u>

I got the idea of trying this construction when J.-C. BERMOND told me how he constructed resolvable decompositions of the complete directed graph into directed 3-circuits. The construction is a modification of the one used by BAKER & WILSON: we use 6 'points at infinity' instead of two. At first it would seem that this requires a NKTS(18) with subdesign NKTS(6) which cannot exist, but on closer inspection one finds that the existence of ingredient c suffices.

Starting with a group divisible design GD(4,1,6;6m) $(X,\mathcal{B},\mathcal{G})$ where X is the point-set ($|X| = 6m$), $\mathcal{B}$ is the collection of blocks (all of size 4) and $\mathcal{G} = \{X_0,X_1,\ldots,X_{m-1}\}$ is the collection of groups (all of size 6) we form a NKTS(12m + 6) with pointset $(X \times \{0,1\}) \cup Z$ as follows:

Let $Z = Y \times \{0,1\}$ where $|Y| = 3$ and take for the parallel class consisting of pairs the collection of pairs $\{(u,0),(u,1)\}$ for $u \in X \cup Y$. We need 6m + 2 parallel classes of triples; for each point $x \in X$ we shall find one parallel class $C_x$ and furthermore we have two other parallel classes $\mathcal{D}$ and $\mathcal{D}'$.

Let $((X_0 \times \{0,1\}) \cup Z, \mathcal{B}_0, \mathcal{G}_0)$ be a NKTS(18) as given by ingredient a, where $\mathcal{G}_0 = \{\{(u,0),(u,1)\} \mid u \in X_0 \cup Y\}$ and $\mathcal{B}_0$ is partitioned into parallel classes $P_i (0 \le i \le 7)$.

Let for $1 \le j \le m-1$ $((X_j \times \{0,1\}) \cup Z, \mathcal{B}_j)$ be a design as given by ingredient c, where each time Z is the block of size 6, $V = (X_j \times \{0,1\}) \cup Z$,

$\{\{(u,0),(u,1)\} \mid u \in X_j\}$ is the parallel class of pairs on $V\backslash Z$, $\mathcal{D}_j$ and $\mathcal{D}'_j$ are the parallel classes of triples on $V\backslash Z$, and $\mathcal{Q}_{j,i}$ $(0 \le i \le 5)$ are the parallel classes of triples on V.

For each block $B \in \mathcal{B}$ construct a GD(3,1,2;8) on $B \times \{0,1\}$ by deleting the point $\infty$ from an AG(2,3) constructed on $(B \times \{0,1\}) \cup \{\infty\}$ in such a way that it contains the lines $\{\infty, (u,0), (u,1)\}$ for $u \in B$. For $u \in B$ let $\mathcal{C}_{u,B}$ denote the pair of the blocks in the GD(3,1,2;8) that were lines parallel to $\{\infty, (u,0), (u,11)\}$ in the AG(2,3).

Now define $\mathcal{C}_x$ as follows:
If $x \in X_0$ then let

$$\mathcal{C}_x = \cup\{\mathcal{C}_{x,B} \mid x \in B \in \mathcal{B}\} \cup \mathcal{P}_{i(x)}$$

and if $x \in X_j$ $(j > 0)$ then let

$$\mathcal{C}_x = \cup\{\mathcal{C}_{x,B} \mid x \in B \in \mathcal{B}\} \cup \mathcal{Q}_{j,i(x)}$$

where for each j we have fixed some bijection $i : X_j \to \{0,1,2,3,4,5\}$.
Finally let

$$\mathcal{D} = \cup\{\mathcal{D}_j \mid j > 0\} \cup \mathcal{P}_6$$

and

$$\mathcal{D}' = \cup\{\mathcal{D}'_j \mid j > 0\} \cup \mathcal{P}_7.$$

This completes the construction of a NKTS(12m + 6).

REMARK: Similar methods work in the case of a RGD(k,1,k-1 ; v) for general k.

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